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An experimental investigation of low Reynolds number secondary streaming effects associated with an oscillating viscous flow in a curved pipe

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This paper deals with nonlinear streaming effects in an oscillating fluid in a curved pipe. The secondary steady velocity field in the cross-sectional plane of the pipe is studied in detail. Our experimental results are compared with the theory of Lyne (1970; that part of his theory which is valid for Reynolds numbers $R_s \ll 1$) and the theory of Zalosh & Nelson (1973). On the basis of these comparisons we conclude that the theories are in practice valid for higher Reynolds numbers R_s than was formally expected.

1. Introduction

It was shown by Lyne (1970) that an oscillating flow along a curved pipe (with circular cross-section) generates a secondary steady streaming in the cross-sectional plane of the pipe. The geometry of the problem is indicated in figure 1. There are three important parameters in the theory of Lyne (1970): $\epsilon^2 = \overline{W}^2/Ra\omega^2$, $\beta = (2\nu/\omega a^2)^{\frac{1}{2}}$ and $R_s = 2\epsilon^2/\beta^2$, where $\overline{W}$ is a typical velocity of the oscillating flow along the curved pipe, R the radius of curvature of the axis of the pipe, a the cross-sectional radius of the pipe, ω the angular frequency of oscillation and ν the kinematic viscosity of the fluid. Lyne (1970) used the method of matched asymptotic expansions to obtain several terms in a perturbation solution for the stream function. The results are valid for $\epsilon \ll 1$, $\beta \ll 1$ and $R_s \ll 1$ or $R_s \gg 1$. Zalosh & Nelson (1973) studied the same problem using another method of solution and their results are valid for arbitrary values of the parameter $\alpha = 2^{\frac{1}{2}}/\beta$, but are restricted to $R_s \ll 1$ only. As far as the steady streaming in the cross-sectional plane is concerned, a related problem has been studied by Kuwahara & Imai (1969). Thus the theoretical studies of this problem have been rather thorough. On the other hand, precise experimental investigations have not yet been carried out, at least to the author's knowledge. (For qualitative observations, see Drinker *et al.* 1969; Lyne 1971; Melrose *et al.* 1972.) Therefore we have carried out a close experimental investigation of the case where $\beta \ll 1$ and $R_s \lesssim 1$ and compared our experimental results with the two theories mentioned above. The agreement is good and only minor discrepancies are observed. These discrepancies can probably be explained by the finite aspect ratio $\delta = a/R = 0{\cdot}1$ in the experiment; in the theories $\delta = 0$.

In order to accomplish the comparison mentioned above, we need to quote

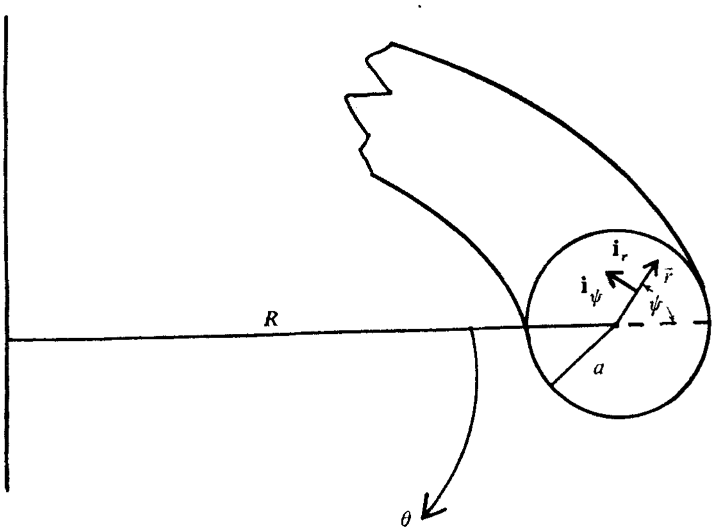


FIGURE 1. The geometry of the problem and the co-ordinate system referred to in the text.

some of the theoretical results obtained by Lyne (1970) and by Zalosh & Nelson (1973). Lyne gave several terms in a matched asymptotic expansion of the stream function. On the basis of some of these terms a uniformly valid expression for the stream function associated with the secondary steady flow in the cross-sectional plane of the pipe may be written in the following way:

$$\begin{aligned}\chi = \beta X_0^{(s)}(\eta,\psi) + \beta^2 X_1^{(s)}(\eta,\psi) + \chi_{00}(r,\psi) + R_s\chi_{01}(r,\psi) + R_s^2\chi_{02}(r,\psi) + \beta\{\chi_{10}(r,\psi) \\ + R_s\chi_{11}(r,\psi) + R_s^2\chi_{12}(r,\psi)\} + \beta^2\{\chi_{20}(r,\psi) + R_s\chi_{21}(r,\psi)\} - M(\eta,\psi;\beta,R_s),\end{aligned} \quad (1)$$

where the functions involved, except $M(\eta,\psi;\beta,R_s)$, are given explicitly in Lyne (1970) as follows: $X_0^{(s)}(\eta,\psi)$ by the steady part of $X_0(\eta,\psi)$, equation (3.26), p. 27; $X_1^{(s)}(\eta,\psi)$ by the steady part of $X_1(\eta,\psi)$, equation (3.41), p. 30; χ_{00}, χ_{01} and χ_{02} by equations (4.4), p. 36, (4.6), p. 37, and (4.8), p. 37, respectively; χ_{10}, χ_{11} and χ_{12} by equations (4.14), p. 38, (4.16), p. 38, and (4.18), p. 38, respectively; χ_{20} and χ_{21} by equation (4.20), p. 39, as

$$\chi_2^{(s)} = \chi_{20} + R_s\chi_{21}. \quad (2)$$

r is the dimensionless radial position $\bar{r}/a$ ($\bar{r}$ and ψ are the radial and angular position, respectively, as indicated on figure 1). η is the scaled variable in the Stokes layer and is defined by (see Lyne 1970)

$$\eta = (1-r)/\beta. \quad (3)$$

The function $M(\eta,\psi;\beta,R_s)$ is the common part of the inner and outer solutions in the matching region and is found to be to the relevant order

$$\begin{aligned}M(\eta,\psi;\beta,R_s) = \beta\{\tfrac{5}{8} - \tfrac{1}{4}\eta\}\sin\psi + \beta^2\{-\tfrac{3}{16} + \tfrac{1}{2}\eta\}\sin\psi \\ + \beta^2\left\{\frac{3}{8}\sin\psi\,\frac{R_s}{768}\sin 2\psi + \frac{R_s^2}{737\,280}\sin\psi\right\}\eta^2.\end{aligned} \quad (4)$$

The stream function given by (1) is dimensionless. The physical, dimensional stream function is

$$\Psi(\bar{r}, \psi) = a^2\epsilon^2\omega\chi(r, \psi). \tag{5}$$

The physical, dimensional radial velocity component U_L is of special interest for the comparison with our experimental data and thus we have from Lyne (1971, equation (2.13), p. 17)

$$U_L(\bar{r}, \psi) = \bar{r}^{-1}\, \partial\Psi(\bar{r}, \psi)/\partial\psi. \tag{6}$$

This definition of $U_L(\bar{r}, \psi)$ serves as the basis for the comparison between the theory of Lyne (1970) and our experimental data.

From the theoretical results of Zalosh & Nelson (1973) we have chosen the simplified expression for the secondary steady stream function given by the steady part of equation (3.12), p. 698, in their paper. This equation gives for the physical, dimensional stream function

$$\Phi(\bar{r}, \phi) = a^2\epsilon^2\alpha^2\omega F_0(r)\cos\phi, \tag{7}$$

where $\phi = -\psi + \frac{1}{2}\pi$ and $F_0(r)$ is defined at the bottom of p. 698 in Zalosh & Nelson (1973). The radial velocity component is

$$U_z(\bar{r}, \phi) = \bar{r}^{-1}\, \partial\Phi(\bar{r}, \phi)/\partial\phi. \tag{8}$$

This definition of the radial velocity component has been used in the comparison between the theory of Zalosh & Nelson and our experimental data.

2. Apparatus and method of observation

The secondary steady streaming in the cross-sectional plane of the curved pipe was studied experimentally using the equipment shown in figure 2. The curved pipe was made by first cutting a circular trace of semicircular cross-section in a Plexiglas slab (methylmethacrylate). The Plexiglas slab was then divided into two equal parts diametrically relative to the circular trace, and finally the two parts were put together to make the curved pipe shown in figure 2. The curved pipe was linked to a pump (see figure 2) and filled with fluid. The pump forced the fluid to oscillate sinusoidally. The motion of the fluid was rendered visible by tracer particles (aluminium powder) and the particles were photographed using stroboscopic illumination synchronized to the frequency of oscillation or an integral part of this. This direct method of studying the motion in the cross-sectional plane requires an approximately undistorted image of this plane on the film. This was achieved by choosing a fluid with nearly the same index of refraction as the Plexiglas, i.e. 1·49. The fluid was a mixture of (1,2)-dibromoethane† and clear lubricating oils. It should be noted that pure (1,2)-dibromoethane dissolved the Plexiglas, but when mixed with at least an equal volume of lubricating oils, we found that this capability was sufficiently suppressed. The experiment could run for a whole day using such fluid mixtures without destroying the Plexiglas tube, but the tube had to be cleaned immediately afterwards. We found that a mixture of approximately 70 g Tellus Special Oil no. 15 (Shell) and 60 g (1,2)-dibromoethane was a suitable fluid for this investigation.

† For an explanation of this notation, see for example Hart & Schuetz (1966, p. 48).

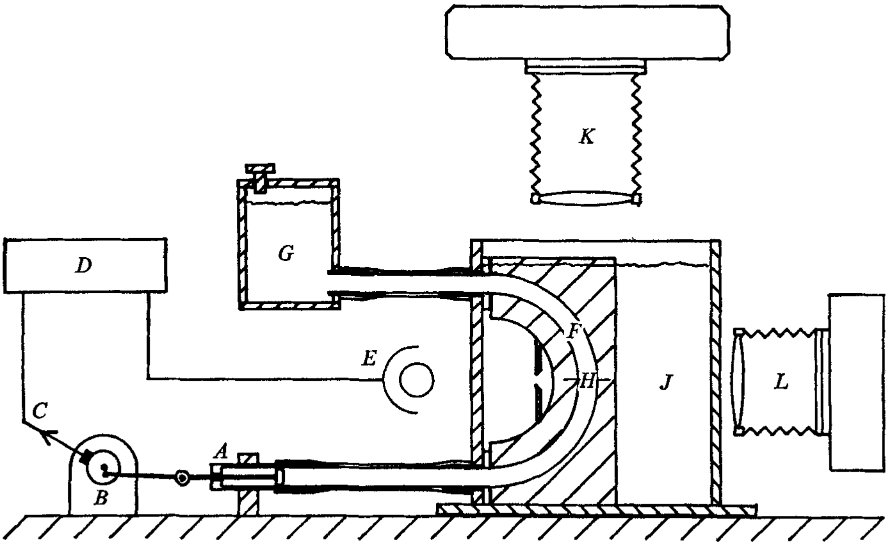


FIGURE 2. Sketch of the apparatus used in the experimental investigation. A, pump; B, motor; C, signal to synchronize the ignition of the stroboscopic lamp to a special phase of the oscillating piston; D, stroboscopic unit; E, stroboscopic lamp; F, curved pipe; G, fluid reservoir; H, observation layer; J, constant-temperature bath; K, camera to observe the secondary streaming; L, camera to observe the amplitude of the oscillations.

The tracks of the tracer particles appear as white lines on the photographs (see figure 3, plate 1). The length of each line can be measured and the exposure time is known, which readily gives the velocity. This requires, of course, that each tracer particle which produces an image on the film does so for the whole exposure time. In the streaming problem that we are concerned with, there is a phase shift in the primary oscillations through the Stokes layer. The repetition of the light flashes from the stroboscopic lamp, however, is phase locked relative to the piston of the pump. This means that particles illuminated in this way can be lost from the field-depth region of the camera when they cross the Stokes layer. The consequence of this is that the radial component of the velocity field in the Stokes layer is scarcely measurable by this technique. But we have obtained other interesting experimental data, which are compared in the next section with the theories of Lyne (1970) and Zalosh & Nelson (1973).

3. Results and discussion

The gross features of the secondary steady streaming in the cross-sectional plane of the curved pipe observed in our experiment are shown in figure 3 (a). In order to get more precise information about the velocity field, we performed several series of measurements where the following quantities were measured.

(a) The thickness δ_{DC} of the vortex systems in the Stokes layer, defined as follows: $\delta_{DC}|_{\psi=0}$ is the distance, measured along the diameter $\psi = 0, \pi$, from the wall of the pipe at $\psi = 0$ to the nearest zero point of the radial velocity; $\delta_{DC}|_{\psi=\pi}$ is the distance, measured along the diameter $\psi = 0, \pi$, from the wall of the pipe at $\psi = \pi$ to the nearest zero point of the radial velocity. Thus we have

$$\delta_{DC}|_{\psi=0} = a - \bar{r}_0|_{\psi=0} \quad \text{and} \quad \delta_{DC}|_{\psi=\pi} = a - \bar{r}_0|_{\psi=\pi},$$

where $\bar{r}_0|_{\psi=n\pi}$, $n = 0, 1$, indicates the zero points mentioned above.

(*b*) The radial velocity component along the diameter $\psi = 0, \pi$.

Values of the most important parameters in the investigation are listed in table 1. From this table it can be seen that we have performed measurements essentially for various β and R_s, because these quantities are important parameters in the theory of Lyne and $\alpha = 2^{\frac{1}{2}}/\beta$ is important in the theory of Zalosh & Nelson.

A comparison between theoretical and experimental values of δ_{DC} is shown in figure 4. From this comparison we see that there is good agreement between our experimental results and the theories for $\beta^{-1} > 15$. For $\beta^{-1} < 15$ we observed $\delta_{DC}|_{\psi=0} < \delta_{DC}|_{\psi=\pi}$, which neither of the theories predicts. This can probably be explained by the finite aspect ratio $\delta = a/r \simeq 0{\cdot}1$ in our experiment; in the theories $\delta = 0$. Another point, however, is that the curved pipe used in the experiment is U-shaped whereas in the theoretical models a toroidal pipe is considered. In spite of this imperfection in the experimental model, we think that the finite aspect ratio is of greatest importance in explaining why $\delta_{DC}|_{\psi=0} < \delta_{DC}|_{\psi=\pi}$. This point is more closely discussed by Bertelsen (1974).

Referring to figure 4, we see that Lyne's theory predicts that the two inner (Stokes-layer) vortex systems occupy the whole cross-sectional plane for $\beta^{-1} \leqslant 9{\cdot}1$ while Zalosh & Nelson's theory gives $\beta^{-1} \leqslant 7{\cdot}4$. It was difficult to observe accurately the transition from two to four vortex systems in the steady streaming in the cross-sectional plane, but certainly for $\beta^{-1} \simeq 8{\cdot}7$ only two vortex systems could be observed and for $\beta^{-1} \simeq 10$ four vortex systems existed. Thus the simplified version of Zalosh & Nelson's theory predicts too low a β^{-1} value for the transition to four-vortex streaming, while Lyne's predictions for this transition are in good agreement with our observations.

Figures 5, 6 and 7 show the radial velocity component as a function of r along the diameter $\psi = 0, \pi$ for various values of β^{-1} and R_s. It can be seen from these figures that the theory of Lyne fits our experimental data nicely, while the simplified theory of Zalosh & Nelson seems to predict velocities which are too high for all parameter values involved in our experimental investigation. More precisely, this theory of Zalosh & Nelson overestimates the radial velocity in the centre of the cross-sectional plane by 25% for $\beta^{-1} \simeq 15{\cdot}5$ and by 10% for $\beta^{-1} \simeq 26{\cdot}2$ (figure 7). This improved agreement for higher β^{-1} values is just what we should expect because the approximation used by Zalosh & Nelson to obtain their simplified theoretical results is based on $\beta^{-1} \gg 1$. We estimate these theoretical results to be practically applicable for $\beta^{-1} \gtrsim 30$. On the other hand, if the more complicated expression for $F_0(r)$ in Zalosh & Nelson's theory (equation (3.6), p. 697) had been used, better agreement would have been obtained for both the thickness δ_{DC} of the inner (Stokes layer) vortex systems and the velocity for lower β^{-1} values as well. In this connexion it could also be relevant to point out that this theory is formally valid for small values of the parameter $(a/R)\,(Ka/\omega\nu)^2$ (see table 1).

Series of measurements	Cross-sectional radius, a (cm)	Radius of curvature R (cm)	Kinematic viscosity ν (cm^2/s)	Frequency of oscillation (Hz)	$\epsilon = s/(aR)^{\frac{1}{2}}$ (s = amplitude of oscillation)	$\beta^{-1} = (\omega a^2/2\nu)^{\frac{1}{2}}$	$R_s = 2\epsilon^2/\beta^2$	$\left(\frac{a}{R}\right)\left(\frac{Ka}{\omega\nu}\right)^2$	Measured quantity: δ_{DC}	Measured quantity: Radial velocity, U
I	0·50	5·00	0·052	5·0	0·0316	8·69	0·151	22·8	×	—
II	0·50	5·00	0·052	7·0	0·0316	10·3	0·212	45·0	×	—
III	0·50	5·00	0·052	10·0	0·0316	12·3	0·302	91·4	×	—
IV	0·50	5·00	0·055	15·0	0·0356	14·6	0·540	230	×	—
V	0·50	5·00	0·055	20·0	0·0356	16·9	0·723	413	×	—
VI	0·50	5·00	0·057	30·0	0·0349	20·3	1·00	824	×	—
VII	0·50	5·00	0·057	50·0	0·0349	26·2	1·67	2290	×	×
VIII	0·54	5·00	0·0475	12·5	0·0316	15·5	0·480	231	×	×
IX	0·54	5·00	0·0475	12·5	0·0480	15·5	1·09	524	×	×
X	0·54	5·00	0·0475	25·0	0·0266	22·0	0·685	663	×	×
XI	0·54	5·00	0·0475	25·0	0·0441	22·0	1·90	1840	×	×

TABLE 1. Important parameter values in the investigation

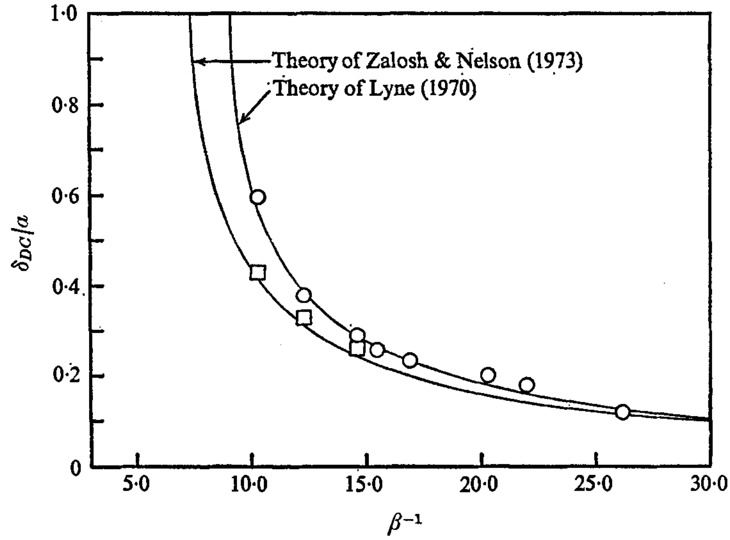


FIGURE 4. The thickness δ_{DC}/a of the vortex systems in the Stokes layer as function of β^{-1}. □, measured values for $\psi = 0$; ○, measured values for $\psi = \pi$.

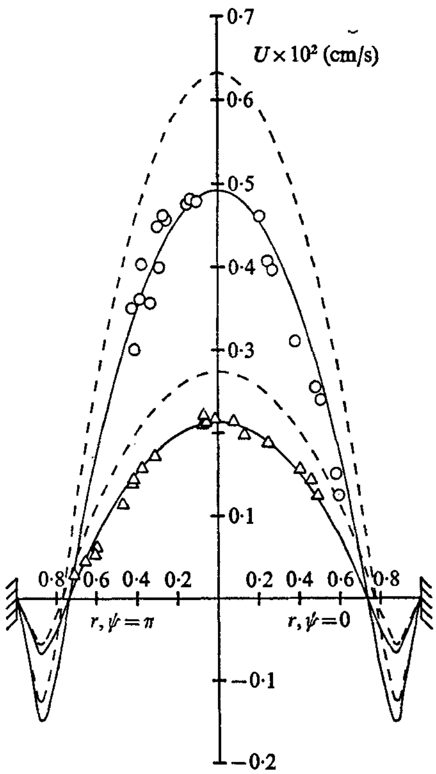


FIGURE 5. The radial component of the secondary steady streaming as a function of r along the diameter $\psi = 0, \pi$. ——, theory of Lyne (1970) as expressed by our equation (6); – – –, asymptotic theory of Zalosh & Nelson (1973) as expressed by our equation (8). The two lower curves were obtained with parameter values as in series VIII, table 1; △, measured velocities in this series. The two upper curves were obtained with parameter values as in series IX, table 1; ○, measured velocities in this series.

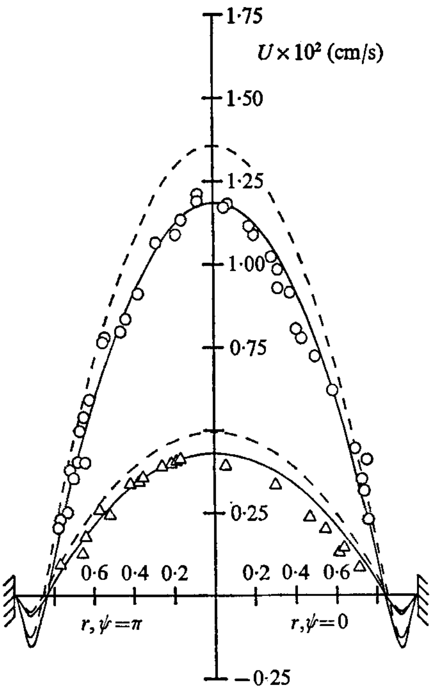


FIGURE 6

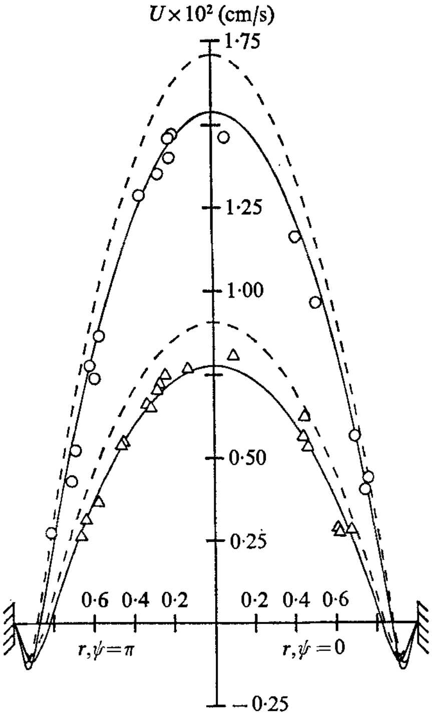


FIGURE 7

FIGURE 6. The radial component of the secondary steady streaming as a function of r along the diameter $\psi = 0, \pi$. ——, theory of Lyne (1970) as expressed by our equation (6); – – –, asymptotic theory of Zalosh & Nelson (1973) as expressed by our equation (8). The two lower curves were obtained with parameter values as in series X, table 1; △, measured velocities in this series. The two upper curves were obtained with parameter values as in series XI, table 1; ○, measured velocities in this series.

FIGURE 7. The radial component of the secondary steady streaming as a function of r along the diameter $\psi = 0, \pi$. ——, theory of Lyne (1970) as expressed by our equation (6); – – –, asymptotic theory of Zalosh & Nelson (1973) as expressed by our equation (8). The two lower curves were obtained with parameter values as in series VI, table 1; △, measured velocities in this series. The two upper curves were obtained with parameter values as in series VII, table 1; ○, measured velocities in this series.

The theories of both Lyne and Zalosh & Nelson are formally valid for $R_s \ll 1$. In our investigation of the radial velocity $0{\cdot}5 \lesssim R_s \lesssim 2$. All curves in figure 5 have $\beta^{-1} \approx 15{\cdot}5$, while $R_s \approx 0{\cdot}48$ on the lower curves and $R_s \approx 1{\cdot}90$ on the upper curves. Corresponding pairs of curves are given in figure 6. On the basis of the results shown in these figures, we conclude that the theory of Lyne (1970) seems to be valid for $R_s \lesssim 2$ provided that $\beta^{-1} \gtrsim 10$. We expect approximately the same region of validity in R_s for the simplified theory of Zalosh & Nelson for $\beta^{-1} \gtrsim 30$. Referring to the theory of Lyne again, we notice that the various terms in the perturbation expansion in R_s (i.e. χ_{00}, χ_{01}, etc.) have favourable numerical coefficients, i.e. all terms are much less than one in the domain $1 \leqslant r \leqslant 0$. This probably explains why Lyne's theoretical model for $R_s \ll 1$ is valid in practice for higher R_s values than was formally expected.

REFERENCES

Bertelsen, A. F. 1974 *Dept. Phys., University of Bergen, Norway, Rep.* no. 67.

Drinker, P. A., Bartlett, R. H., Bialer, R. H. & Noyes, B. S. 1969 *Surgery*, **66**, 775–781.

Hart, H. & Schuetz, R. D. 1966 *Organic Chemistry*, 3rd edn. Boston: Houghton Mifflin.

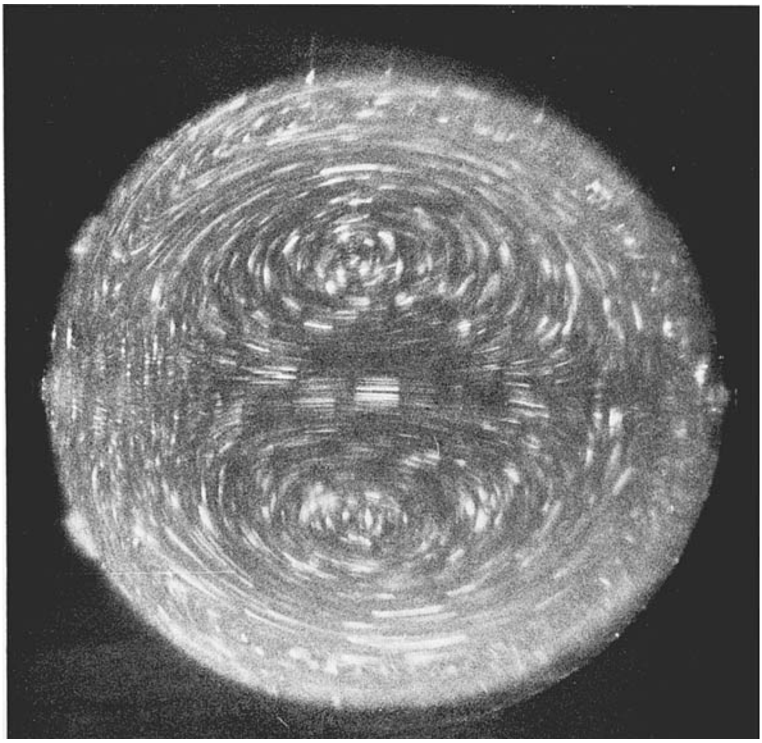
Kuwahara, K. & Imai, I. 1969 *Phys. Fluids Suppl.* **12**, II 94–101.

Lyne, W. H. 1970 Ph.D. thesis, University of London.

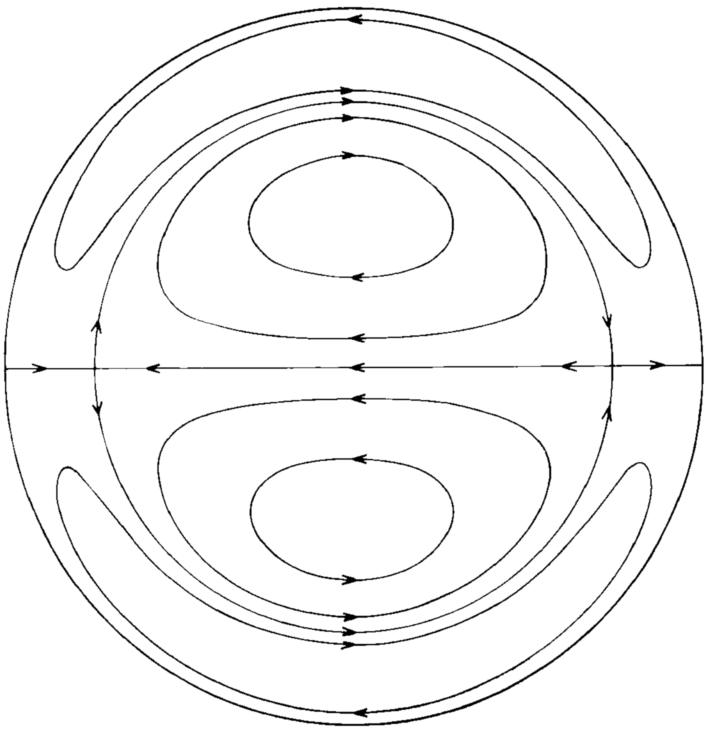
Lyne, W. H. 1971 *J. Fluid Mech.* **45**, 13–31.

Melrose, D. G., Burns, N., Singh, M. P., Elliott, R. L., Read, R., Williams, F. E., Becket, J., Lamb, M. P. & Adams, J. S. 1972 *Biomed. Engng*, **7**, 60–66.

Zalosh, R. G. & Nelson, W. G. 1973 *J. Fluid Mech.* **59**, 693–705.



(*a*)



(*b*)

FIGURE 3. (*a*) Gross features of the secondary steady streaming in the cross-sectional plane of the pipe as observed in our experimental investigation, series IX, table 1. (*b*) Streamline diagram from the theory of Lyne (1970) with parameter values as in (*a*).